JOURNAL OF COMPUTATIONAL PHYSICS 29, 289–294 (1978)

Note

Smooth Perturbations of the Schrödinger Equation Related to the Charmonium Models

1. Introduction

In recent years many publications have appeared where the masses of some elementary particles have been introduced through the bound states of a Schrödinger type equation. This approach has been especially developed in the case of the ψ-particles or charmoniums [5, 6] which were discovered in the experiments with electron–positron annihilation and were interpreted as bound states of a charm–anticharm quark pair. Different types of model potentials have been used but the general feature of all these potentials is that in the case of s-states they behave linearly in the vicinity of the origin.

In this article we try to answer the following question. If the potential is a smooth perturbation of a linear potential, what is the perturbation series for the energy? More strictly speaking we assume that in the vicinity of the origin the potential $V(r)$ can be expanded in a Taylor series of the form

$$V(r) = cr + \epsilon v_2 r^2 + \epsilon^2 v_3 r^3 + ..., \tag{1}$$

where $\epsilon \ll 1$ and the other coefficients c, v_2, v_3 are of the order of magnitude of unity so that subsequent terms of the series (1) are of less importance than the previous ones. One gets the answer in the form

$$E_n = E_n^{(0)}(c) + \epsilon E_n^{(1)}(c, v_2) + \epsilon^2 E_n^{(2)}(c, v_2, v_3) \cdots \tag{2}$$

The introduction of the parameter ϵ can be considered as an artificial way to obtain the final result so the actual validity of the expansion (2) depends upon how succeeding terms decrease.

Of course more complicated perturbations can be treated, e.g., the perturbations that lead to the loss of quarks confinement [9, 10]. In order to estimate their contribution other techniques than presented here are needed.

The final results can be obtained with the help of the well-known Schrödinger perturbation theory using the known matrix elements [4] but we shall present here a more efficient method proposed by one of the authors (S. Yu. Slavyanov [11]). The advantage of this method is that it can be easily programmed with the help of algebraic computing systems (ACS for short). We have used for our computations two ACS: SYMBAL [3] and REDUCE [8].

0021-9991/78/0292–0289$02.00/0

The article by Barton and Fitch [2] can be recommended as a good introduction to the use of algebraic computing in theoretical physics.

2. Obtaining the Expansion for Energy

We shall start with the following equation (cf. [6]) for the radial part $u(r)$ ($\Psi(r) = Y_{00}(\vartheta, \varphi) \cdot u(r)/r$) of the wave function $\Psi(r)$ for s-states

$$u''(r) + m[E - V(r, \epsilon)]u(r) = 0 \tag{3}$$

and the boundary conditions

$$u(r)|_{r=0} = 0, \qquad u(r) \to 0 \quad r \to \infty. \tag{4}$$

Here m is the mass of the charmed quark and the potential $V(r, \epsilon)$ is represented by (1) in the vicinity of the origin and increases to infinity provided the confinement of quarks is fulfilled. The charmonium masses spectrum in the nonrelativistic limit is

$$M_n = 2m + E_n .$$

After introducing a new scale $x = \epsilon r$, a new "large" parameter $\nu = m^{1/3}c^{1/3}\epsilon^{-1}$ ($\nu \gg 1$) and a new energy $\lambda = m^{1/3}c^{-2/3}E$, Eq. (3) can be rewritten in the form

$$u''(x) + [\nu^2\lambda - \nu^3 v(x)]u(x) = 0, \tag{6}$$

where

$$v(x) = x + v_2/cx^2 + v_3/cx^3 + \dots .$$

According to the modification of the comparison equation method proposed by one of the authors [11] we take the solutions $u(x)$ to be of the form

$$u_n(x) = [z'(x, \nu)]^{-1/2} Ai(\nu \cdot z(x, \nu) - \mu_n). \tag{7}$$

Here $Ai(t)$ is one of the standard Airy functions and μ_n are of subsequent absolute values of roots of $Ai(t)$ ($\mu_0 = 2.3381$, $\mu_1 = 4.0879$, $\mu_2 = 5.5210$, $\mu_3 = 6.7867. \dots$). In other words μ_n are the eigenvalues of the Schrödjnger equation with a linear potential. Substituting the solution (7) into Eq. (6) one gets the following equation for the function $z(x, \nu)$ which determines the nonlinear transformation of scale

$$z'^2(z - v(x)) - (1/\nu)(z'^2\mu_n - \lambda_n) - (1/\nu^3)\{z, x\} = 0, \tag{8}$$

where $\{z, x\}$ denotes the schwarzian derivative

$$\{z, x\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2. \tag{9}$$

From the boundary condition (4) it follows that

$$z(x, \nu)|_{x=0} = 0, \tag{10}$$

and Eq. (8) and condition (10) together give the expression for the eigenvalue λ_n

$$\lambda_n = \mu_n z'^2 |_{x=0} + \frac{1}{2\nu^3} \{z, x\}\Big|_{x=0}, \tag{11}$$

which means actually that the values of energy are determined by the Taylor expansion of the function $z(x)$ in the vicinity of the origin.

Now we shall expand the function $z(x, \nu)$ in the series

$$z(x, \nu) = \sum_{k=0}^{\infty} z_k(x)\, \nu^{-k} \tag{12}$$

and the eigenvalues λ_n in the series

$$\lambda_n = \sum_{k=0}^{\infty} \lambda_n^{(k)} \nu^{-k}. \tag{13}$$

If we substitute expansions (12) and (13) into Eq. (8) and equate the subsequent negative powers of ν we get a system of linear equations for the coefficients $z_i(x)$ that together with the initial condition (10) give the following expressions for $z_i(x)$

$$\begin{aligned}
z_0(x) &= \left[\frac{2}{3} \int_0^x [v(x)]^{1/2}\, dx\right]^{3/2}, \\
z_1(x) &= \frac{1}{z_0^{1/2}} \int_0^x \frac{\mu z_0^2 - \lambda^{(0)}}{2[v(x)]^{1/2}}\, dx, \\
z_n(x) &= \frac{1}{z_0^{1/2}} \int_0^x \frac{1}{2[v(x)]^{1/2}} \Big[\mu \sum_{k=0}^{n-1} z_k' z_{n-k-1}' - \lambda^{(n-1)} \\
&\quad + z_0' \sum_{k=1}^{n-1} z_k' z_{n-k} + \sum_{k=1}^{n-1} z_k' \sum_{i=0}^{n-k} z_i' z_{n-k-i} + \frac{1}{2} \{z, x\}_{n-3}\Big]\, dx,
\end{aligned} \tag{14}$$

where by $\{z, x\}_k$ we denote the coefficients by ν^{-k} after substituting the expansion (12) into the Schwarzian derivative (9). In order to obtain N terms in expansion (13) it is not necessary to have the exact expressions for all $z_i(x)$ but we have to obtain $N + 1$ terms in the Taylor expansion for $z_0(x)$, N terms for $z_1(x)$ and so on. Therefore the procedure reduces to handling truncated power series. It has been programmed in SYMBAL and took approximately half a minute of computing time on a CDC 6400 to obtain the final expression for energy but the computations took a large amount of memory (about 120 k-words).

The more straightforward way for doing the calculations is to take the function $z(x, \nu)$ to be of the form

$$z(x, \nu) = \sum_{k=0}^{N} \nu^{-k} \sum_{i=1}^{N+2-k} a_{ki} x^i, \tag{15}$$

then substitute expansions (15) and (13) into eq. (8) and equate both the powers of ν and x. The second approach was programmed in REDUCE and took about 7 min of computing time on DEC 10 and a large amount of memory as well. Both methods gave the same answer which gives confidence in the accuracy of the result.

Here we give the obtained expansion for energy E in terms of the initial parameters

$$\begin{aligned} E_n = c^{2/3} m^{-1/3} \Big\{ & \mu_n + \epsilon \frac{8}{15} \mu_n^2 \frac{v_2}{c} \\ & + \epsilon^2 \left[\mu_n^3 \left(\frac{16}{35} \frac{v_3}{c} - \frac{48}{175} \frac{v_2^2}{c^2} \right) + \left(\frac{3}{7} \frac{v_3}{c} - \frac{9}{35} \frac{v_2^2}{c^2} \right) \right. \\ & + \epsilon^3 \left[\mu_n^4 \left(\frac{128}{315} \frac{v_4}{c} - \frac{1088}{1575} \frac{v_2 v_3}{c^2} + \frac{22912}{70875} \frac{v_2^3}{c^3} \right) \right. \\ & \left. + \mu_n \left(\frac{80}{63} \frac{v_4}{c} - \frac{596}{315} \frac{v_2 v_3}{c^2} + \frac{1336}{1575} \frac{v_2^3}{c^3} \right) \right] \\ & + \epsilon^4 \left[\mu_n^5 \left(\frac{256}{693} \frac{v_5}{c} - \frac{41984}{51975} \frac{v_2 v_4}{c^2} - \frac{17344}{40425} \frac{v_3^2}{c^2} \right. \right. \\ & \left. + \frac{2507264}{1819125} \frac{v_2^2 v_3}{c^3} - \frac{41575168}{81860625} \frac{v_2^4}{c^4} \right) \\ & + \mu_n^2 \left(\frac{1808}{693} \frac{v_5}{c} - \frac{50432}{10395} \frac{v_2 v_4}{c^2} - \frac{19324}{8085} \frac{v_3^2}{c^2} \right. \\ & \left. + \frac{536512}{72765} \frac{v_2^2 v_3}{c^3} - \frac{942752}{363825} \frac{v_2^4}{c^4} \right) + O(\epsilon^5) \Big\}. \end{aligned} \tag{16}$$

In order to make it easier to estimate the terms we give here also the approximate expression to five decimal places

$$\begin{aligned} E_n = c^{2/3} m^{-1/3} \{ & \mu_n + \epsilon \mu_n^2 \cdot 0.53333 v_2/c \\ & + \epsilon^2 [\mu_n^3 (0.45714 v_3/c - 0.27429 v_2^2/c^2) \\ & + (0.42857 v_3/c - 0.25714 v_2^2/c^2)] + \epsilon^3 [\mu_n^4 (0.40635 v_4/c \\ & - 0.69079 v_2 v_3/c^2 + 0.32327 v_2^3/c^3) + \mu_n (1.26984 v_4/c \\ & - 1.89206 v_2 v_3/c^2 + 0.84825 v_2^3/c^3)] + \epsilon^4 [\mu_n^5 (0.36941 v_5/c \\ & - 0.80777 v_2 v_4/c^2 - 0.42904 v_3^2/c^2 + 1.37828 v_2^2 v_3/c^3 \\ & - 0.50788 v_2^4/c^4) + \mu_n^2 (2.60894 v_5/c - 4.85157 v_2 v_4/c^2 \\ & - 2.39010 v_3^2/c^2 + 7.37322 v_2^2 v_3/c^3 - 2.59122 v_2^4/c^4)] \\ & + O(\epsilon^5) \}. \end{aligned} \tag{17}$$

From formula (17) one can observe that the obtained expansion is generally valid for calculations when $\epsilon\mu_n \leqslant \frac{1}{2}$, but of course this depends also upon the coefficients c, v_2, v_3, v_4, and v_5.

3. Some Examples and Speculations

As an example for doing a calculation with expansion (16) we take the potential

$$V(r) = \frac{1}{\epsilon(\epsilon r + 1)} - \frac{1}{\epsilon}, \tag{18}$$

which appeared as a solution of Einstein–Maxwell equations in relativity theory [7]. Here $c = v_3 = v_5 = 1$, $v_2 = v_4 = -1$, and we get for the energy

$$\begin{aligned} E_n = c^{2/3} \Big[\mu_n &- \epsilon \frac{8}{15} \mu_n{}^2 + \epsilon^2 \left(\frac{32}{175} \mu_n{}^3 + \frac{6}{35} \right) \\ &+ \epsilon^3 \left(\frac{43072}{70875} \mu_n{}^4 + \frac{772}{525} \mu_n \right) \\ &+ \epsilon^4 \left(\frac{245312}{81860625} \mu_n{}^5 + \frac{54308}{363825} \mu_n{}^2 \right) + O(\epsilon^5) \Big]. \end{aligned} \tag{19}$$

It is well known that the first-order perturbation approximation for the energy is given by the matrix elements. The formula (16) therefore gives one the opportunity to obtain the matrix elements of powers of r for the linear potential

$$\begin{aligned} \langle r^2 \rangle &= \frac{8}{15} \mu_n{}^2, \\ \langle r^3 \rangle &= \frac{16}{35} \mu_n{}^3 + \frac{3}{7}, \\ \langle r^4 \rangle &= \frac{128}{315} \mu_n{}^4 + \frac{80}{63} \mu_n. \\ \langle r^5 \rangle &= \frac{256}{693} \mu_n{}^5 + \frac{1808}{693} \mu_n{}^2. \end{aligned} \tag{20}$$

This corresponds to the results of previous authors [4].

Our calculations also give the possibility of obtaining another important quantity

$$D = \frac{|\, u'(0)|^2}{\int_0^\infty u^2\, dr}, \tag{21}$$

which is essential for determining the lifetime of the quark–antiquark pairs [1]. According to representation (7) one can rewrite this quantity as

$$D = \frac{|\, Ai'^2(-\mu_n)|^2 \cdot z'(0, \nu)}{\int_0^\infty Ai^2(-\mu_n + z)(x'(z, \nu))^2\, dz}, \tag{22}$$

where $x(z, \nu)$ is the inverse function of $z(x, \nu)$. Using the well-known relation

$$\frac{Ai'^2(-\mu_n)}{\int_0^\infty Ai^2(-\mu_n + x)\,dx} = 1 \tag{23}$$

and also the derived matrix elements (20) one can finally get

$$D = 1 + \frac{1}{3}\,\epsilon\mu_n\,\frac{v_2}{c} + \epsilon^2\mu_n{}^2\left(\frac{8}{5}\,\frac{v_3}{c} - \frac{224}{225}\,v_2{}^2/c^2\right) + O(\epsilon^3). \tag{24}$$

As yet we have not tried to compare the above theoretical results with experimental data because of the paucity of the latter. But a comparison should be possible as more data become available.

Acknowledgments

This work has been supported by the Visitor Exchange Program between the universities of Leningrad and Stockholm. One of the authors (S. Yu. Slavyanov) would like to express his gratitude to the University of Stockholm, Institute of Theoretical Physics for its hospitality and also to Bo Thidv̄ for useful discussions.

References

1. T. Appelquist and H. D. Politzer, *Phys. Rev. Lett.* **34** (1975), 43–45.
2. D. Barton and T. P. Fitch, *Rep. Progr. Phys.* **35** (1972), 235–314.
3. M. E. Engeli, *Advan. Inform. Systems Studies* **1** (1969), 117–171.
4. P. L. Ferreira and J. A. C. Alcaras, *Lett. Nuovo Cimento* **14** (1975), 500–504.
5. R. C. Giles and S.-H. H. Tye, *Phys. Rev. Lett.* **34** (1976), 168–171.
6. B. J. Harrington, S. Y. Park, and A. Yildiz, *Phys. Rev. Lett.* **37** (1975), 1175–1178.
7. M. Hessaby, "Theory of Infinitely Extended Particles," Tehran Univ. Press, Tehran, Iran, 1977.
8. A. C. Hearn *in* "Proceedings of the 2nd Symposium on Symbolic and Algebraic Manipulations, New York, 1971."
9. J. Kogut and L. Sussking, *Phys. Rev. D* **12** (1975), 2821–2843.
10. H. J. W. Müller-Kirsten and N. Vahedi, *Phys. Rev. D* **14** (1976), 3079–3085.
11. S. Yu. Slavyanov, *Differencialnye Uravneniya* **5** (1969), 313–325, "Topics in Mathematical Physics," Vol. 4, Consultants Bureau, New York, 1971.

Received: December 28, 1977

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